An Exact Method for Calculating the Force on the GRS Proof Mass from Stray Charges

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Motivation and Objective

- To develop an accurate measure via analytic methods of the force exerted between various GRS proof mass and housing wall geometries by stray charges and patch effects.
- To avoid inaccuracies inherent in numerical techniques for extreme aspect ratio cases.

Technique

- Specify the governing PDE and boundary conditions.
- Decompose domain into rectangles.
- Obtain the analytic solution on each rectangle by separation of variables.
- Enforce compatibility conditions across common boundaries.

Calculating the Force on the GRS Proof Mass from Stray Charges in the Coating Gaps

- **Model the coating gap cross-section as a T-shaped domain:**

$$\Omega = \Omega_1 \cup \Omega_2 \cup (-s,s)\times\{0\}, \text{ where } \begin{cases} \Omega_1 = (-w,w)\times(0,h) \\ \Omega_2 = (-s,s)\times(-d,0) \end{cases}$$

- **Solve the associated potential problem:**

$$\begin{cases} \Delta\phi(x,y) = \phi_{xx}(x,y)+\phi_{yy}(x,y) = 0,(x,y)\in\Omega \\ \phi(x,h) = 0, x\in(-w,w) \\ \phi_x(\pm w,y) = 0, y\in(0,h) \text{ (green boundary)} \\ \phi(x,0) = 0, x\in(-w,-s)\cup(s,w) \\ \phi(\pm s,y) = 0, y\in(-d,0) \\ \phi(x,-d) = f(x), x\in(-s,s) \text{ (red boundary)} \end{cases}$$

- **The analytic solution is given by:**

$$\phi_1(x,y) = C_o(y-h) + \sum_{n=1}^{\infty} C_n\left[e^{n\pi\frac{y}{w}} - e^{n\pi\frac{(2h-y)}{w}}\right]\cos\left(n\pi\frac{x}{w}\right)$$

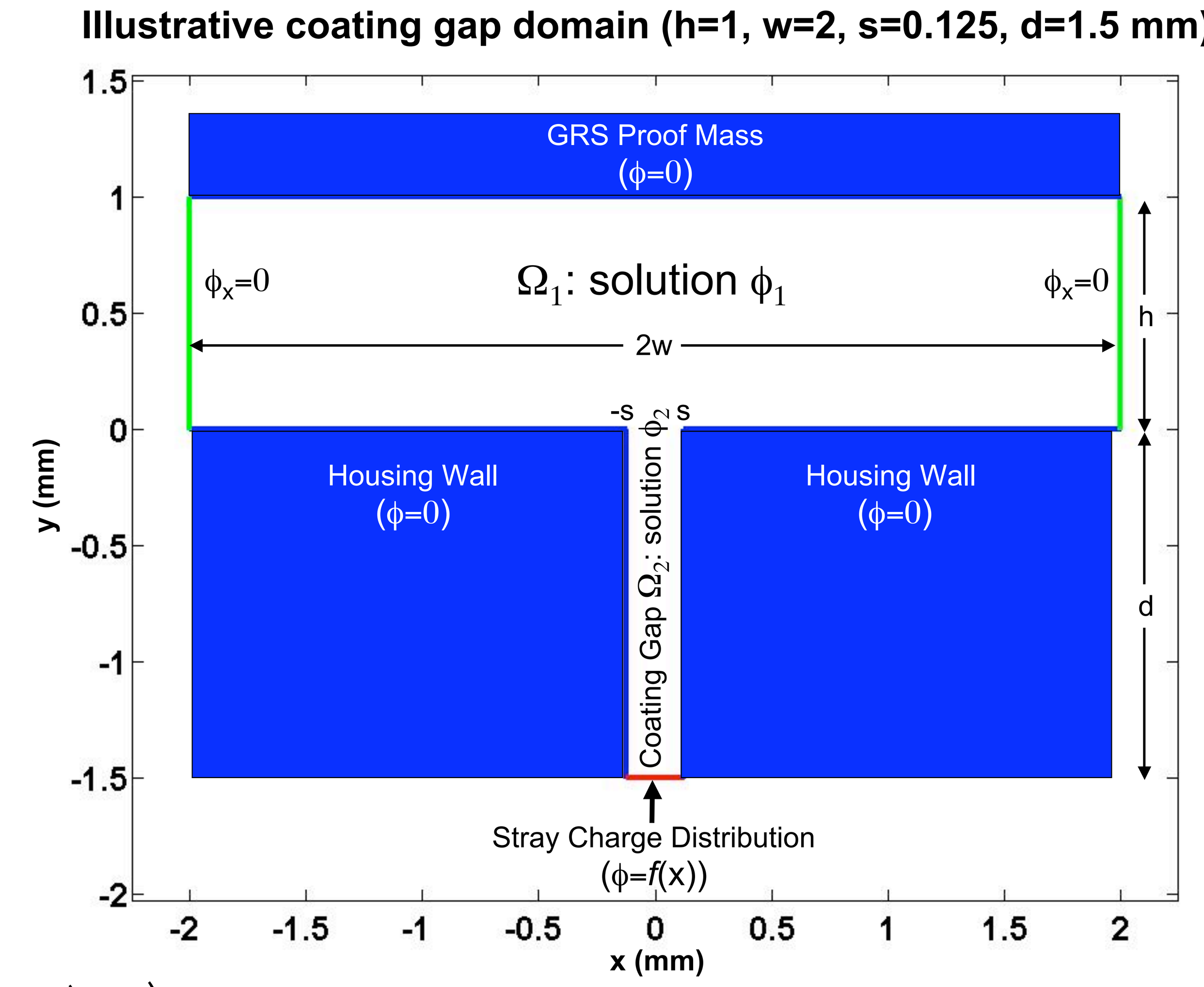
$$\phi_2(x,y) = \sum_{n=1}^{\infty}\left[\left(A_n - B_n e^{(2n-1)\pi\frac{d}{s}}\right)e^{\left(n-\frac{1}{2}\right)\pi\frac{y}{s}} + (B_n - A_n)e^{-\left(n-\frac{1}{2}\right)\pi\frac{y}{s}}\right]\cos\left(\left(n-\frac{1}{2}\right)\pi\frac{x}{s}\right)$$

where the Fourier coefficients A_n, B_n, and C_n are determined by:

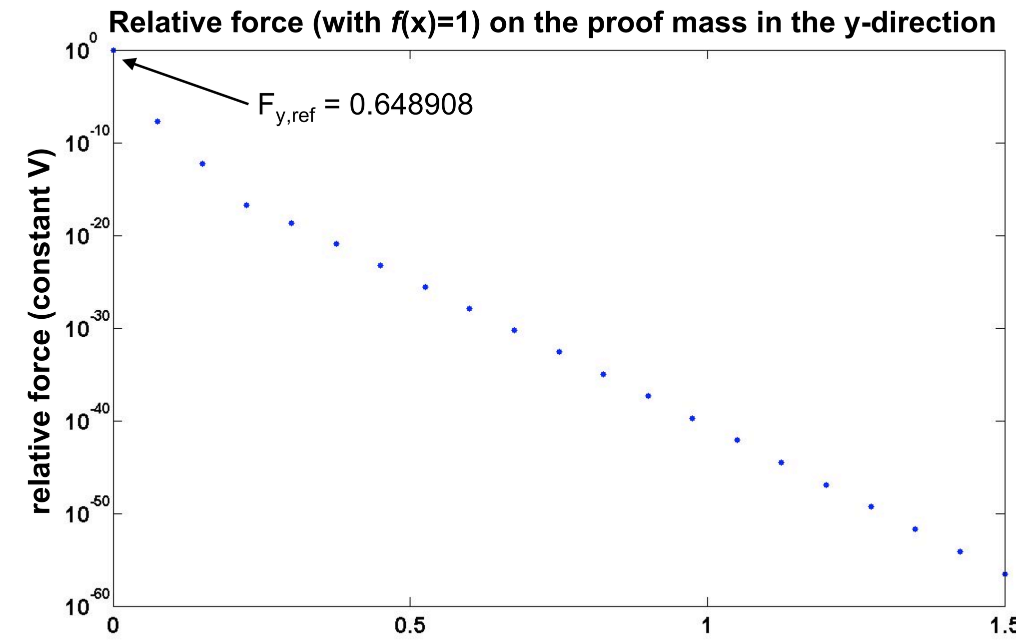
- **orthogonality relations,**
- **boundary conditions, and**
- **the compatibility conditions** $\begin{cases} \phi_1(x,0) = \phi_2(x,0) \\ \phi_{1y}(x,0) = \phi_{2y}(x,0) \end{cases}$ **for** $x\in(-s,s)$.

- **Then** $\phi|_{\Omega_1} = \phi_1, \phi|_{\Omega_2} = \phi_2$, **and** $\phi(x,0) = \phi_1(x,0) = \phi_2(x,0)$ **for** $x\in(-s,s)$.

Illustrative coating gap domain (h=1, w=2, s=0.125, d=1.5 mm)



Relative force (with f(x)=1) on the proof mass in the y-direction

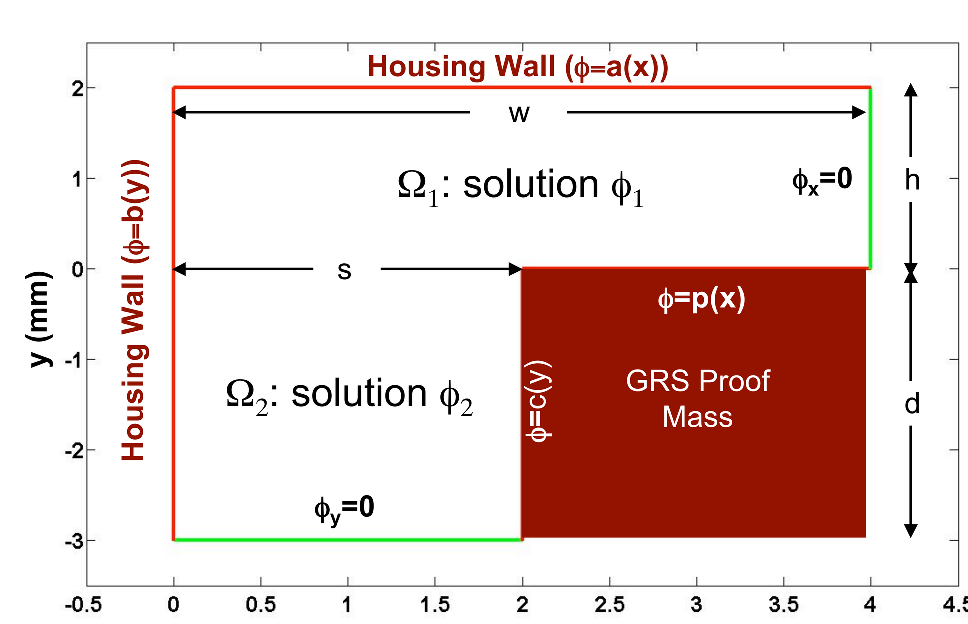


Calculating the Force on the GRS Proof Mass at the Corners from Patch Effects

- **As above, with** $\Omega = \Omega_1 \cup \Omega_2 \cup (0,s)\times\{0\}$, **where** $\begin{cases} \Omega_1 = (0,w)\times(0,h) \\ \Omega_2 = (0,s)\times(-d,0) \end{cases}$

- **Solve the associated potential problem:**

$$\begin{cases} \Delta\phi(x,y) = \phi_{xx}(x,y)+\phi_{yy}(x,y) = 0,(x,y)\in\Omega \\ \phi(x,h) = a(x), x\in(0,w) \\ \phi(0,y) = b(y), y\in(-d,h) \\ \phi_y(x,-d) = 0, x\in(0,s) \text{ (green boundary)} \\ \phi(s,y) = c(y), y\in(-d,0) \\ \phi(x,0) = p(x), x\in(s,w) \\ \phi_x(w,y) = 0, y\in(0,h) \text{ (green boundary)} \end{cases}$$



- **The analytic solution:**

$$\begin{cases} \phi_1(x,y) = \sum_{n=1}^{\infty}\left\{\left[A_n\left(e^{\left(n-\frac{1}{2}\right)\pi\frac{y}{w}} - e^{-\left(n-\frac{1}{2}\right)\pi\frac{y}{w}}\right) + C_n\left(e^{\left(n-\frac{1}{2}\right)\pi\frac{y}{w}} - e^{\left(n-\frac{1}{2}\right)\pi\frac{2h-y}{w}}\right)\right]\sin\left(\left(n-\frac{1}{2}\right)\pi\frac{x}{w}\right) + B_n\left(e^{n\pi\frac{x}{h}} + e^{n\pi\frac{2w-x}{h}}\right)\sin\left(n\pi\frac{y}{h}\right)\right\} \\ \phi_2(x,y) = \sum_{n=1}^{\infty}\left\{\left[E_n\left(e^{-\left(n-\frac{1}{2}\right)\pi\frac{x}{d}} - e^{\left(n-\frac{1}{2}\right)\pi\frac{x-2s}{d}}\right) + F_n\left(e^{\left(n-\frac{1}{2}\right)\pi\frac{x}{d}} - e^{-\left(n-\frac{1}{2}\right)\pi\frac{x}{d}}\right)\right]\sin\left(\left(n-\frac{1}{2}\right)\pi\frac{y}{d}\right) + D_n\left(e^{n\pi\frac{y}{s}} + e^{-n\pi\frac{y+2d}{s}}\right)\sin\left(n\pi\frac{x}{s}\right)\right\} \end{cases}$$

where the Fourier coefficients A_n, B_n, C_n, D_n, E_n, and F_n are determined as above.

- **Then** $\phi|_{\Omega_1} = \phi_1, \phi|_{\Omega_2} = \phi_2$, **and** $\phi(x,0) = \phi_1(x,0) = \phi_2(x,0)$ **for** $x\in(0,s)$.

Conclusion

- This yields an exact method for calculating the effects of piecewise-continuous (surface) stray charges or patch effects in multiple connected rectangular domains, independent of inaccuracies inherent in numerical techniques for extreme aspect ratio cases.
- This method can be applied to cases with surface irregularities: specify boundary conditions above as the *effective* potential distribution.